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LETTER TO THE EDITOR

Fractional statistics and the Z_3 Potts model

P Bhattacharyya and Spenta R Wadia
Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India

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Abstract. We construct a generalised Jordan-Wigner transformation which converts the one-dimensional quantum Z_3 Potts Hamiltonian into one involving fermions obeying fractional statistics. A naive, scale-invariant continuum limit of the discrete model is taken.

There has been a recent resurgence of interest in 2D statistical mechanical systems which has come about because of the use of conformal invariance in classifying critical phenomena in such systems (Polyakov 1970, Belavin *et al* 1984, Friedan *et al* 1984). It is well known that at a critical transition, where the correlation length diverges, an underlying continuum theory can be associated (Kogut 1979) with the system. For instance, as shown by Ferrell (1973), corresponding to the 2D Ising model there exists a free 1D Dirac equation such that the free energy per site of one is the ground state energy of the other. It is therefore of considerable interest to construct continuum theories which fit this classification scheme. Furthermore, in 1D the Lorentz spin is not restricted to take on only integral or half-integral values but can be continuously varying. Particles obeying fractional statistics in two space dimensions have also been invoked in the explanation of the fractional quantised Hall effect (Halperin 1984). The work described in this letter was motivated not only for a search of models to fit this classification scheme but also to explore these new degrees of freedom.

For this purpose, we study the Z_3 Potts model (Wu 1982) and construct the underlying continuum theory which emerges at criticality. The model is of importance in its own right as it has been solved only at criticality and, according to Baxter (1982), the only hope of solving it away from criticality might be through a generalisation of algebraic methods (Schultz *et al* 1964) similar to that used by Onsager for solving the zero-field 2D Ising model. Such a generalisation is also carried out here. The anisotropic Z_3 Potts Hamiltonian is defined by

$$H = -3\beta_x \sum_i \delta_{n_i, n_{i+\hat{x}}} - 3\beta_y \sum_i \delta_{n_i, n_{i+\hat{y}}}, \qquad n_1 = 0, 1, 2. \tag{1}$$

Consider first the one-dimensional case. The transfer matrix is given by

$$T = e^{2\beta}[1 + e^{-3\beta}(P + P^\dagger)],$$

where $P|n\rangle = |n+1\rangle$, $P^3 = 1$. We can define a dual coupling $\tilde{\beta}$ through

$$\exp[\tilde{\beta}(P + P^\dagger)] = f(\tilde{\beta}) + g(\tilde{\beta})(P + P^\dagger), \tag{2}$$

where $f(\tilde{\beta})$ and $g(\tilde{\beta})$ can be explicitly calculated. Duality demands that

$$\exp(-3\beta) = g(\tilde{\beta})/f(\tilde{\beta}) = [\exp(3\tilde{\beta}) - 1]/[\exp(3\tilde{\beta}) + 2] \tag{3}$$

which is the usual relation obtained by graphical methods (Wu 1982). In two dimensions, the transfer matrix of the model can be written in the limit of extremely anisotropic coupling constants (Kogut 1979) as

$$T = \exp(-H), \tag{4}$$

where

$$H = -\lambda \sum_x (Q_x Q_{x+1}^\dagger + \text{HC}) - \sum_x (P_x + P_x^\dagger). \tag{5}$$

In this limit, the contribution arising out of the non-commutativity of the two terms can be neglected. The self-duality of the Hamiltonian (5) can be made manifest in exact analogy with the Ising case. The critical transition takes place at $\lambda = 1$.

From our definitions of the operators P and Q, we see that they satisfy the algebra $Q_x P_x = z P_x Q_x$ ($z \equiv \exp(2\pi i/3)$) on the same site. On different sites they commute. The key step now is to construct the analogue of the Jordan–Wigner transformation which, for the Ising model, transforms the Pauli matrices into fermions. In order to do so here, we proceed in close analogy with the Ising case.

It is most convenient to proceed in two stages. Following Itzykson (1982), we first construct a set of unitary operators $\{\Gamma_\alpha(x)\}$ ($\alpha = 1$-3) such that $\Gamma_\alpha^3(x) = 1$. Explicitly, we have

$$\Gamma_1(x) = \left(\prod^{x-1} P_k\right) Q_x, \qquad \Gamma_2(x) = \left(\prod^{x-1} P_k\right) Q_x P_x, \tag{6}$$

$$\Gamma_3(x) = \left(\prod^{x-1} P_x\right) Q_x P_x^\dagger.$$

In terms of these operators the 1D quantum Hamiltonian can be written as

$$H = -\lambda \sum (\Gamma_2(x)\Gamma_1^\dagger(x+1) + \Gamma_1(x+1)\Gamma_2^\dagger(x)) - \sum (\Gamma_2^\dagger(x)\Gamma_1(x) + \Gamma_1^\dagger(x)\Gamma_2(x)). \tag{7}$$

We next make a canonical transformation which takes Q to $P^\dagger$ and P to Q. The usual Jordan–Wigner transformation can now be generalised in two different ways. In the first route (Alcaraz and Köberle 1981), the operators $\{\Gamma_\alpha(x)\}$ can be expressed in terms of parafermions (as defined by Fradkin and Kadanoff (1980)) which obey the relation $C^3(x) = 0$. This, however, has the difficulty that the Hamiltonian given by (6) immediately becomes nonlinear and it is not obvious how to proceed further. We follow an alternative route in which $\{\Gamma_\alpha(x)\}$ can be represented in terms of objects which obey the Pauli principle (i.e. $C^2(x) = 0$) but have an algebra different from the usual fermionic one. These are given by

$$C_\alpha(x) = \left(\prod^{x-1} Q_k\right) M_\alpha(x) \tag{8a}$$

where

$$3M_1 = P^\dagger(1 + Q + Q^\dagger), \qquad 3M_2 = P^\dagger(1 + z^* Q + z Q^\dagger),$$

$$3M_3 = P^\dagger(1 + zQ + z^* Q^\dagger). \tag{8b}$$

In terms of these, we have

$$\Gamma_1(x) = \sum_{\alpha=1}^{3} C_\alpha(x), \qquad \Gamma_2(x) = \sum_{\alpha=1}^{3} z^{\alpha-1} C_\alpha(x). \tag{9}$$

The Ising Hamiltonian is recovered by inserting (9) into (6) with $z=-1$ and noting that $C_2(x)=C_1^\dagger(x)$. Corresponding to this last property, the operators $C_\alpha(x)$ satisfy $C_2(x)C_1(x)=C_3^\dagger(x)$ and cyclic permutations. Also $C_1(x)C_2(x)=0$. As a result of this property only two of the variables are independent. The algebra satisfied by these operators can be easily obtained from their definitions in terms of P and Q:

$$C_\alpha^\dagger(x)C_\beta(y)=(z\theta(x-y)+z^*\theta(y-x))C_\beta(y)C_\alpha^\dagger(x), \qquad x\neq y. \tag{10}$$

Using the various properties of the operators $C_\alpha(x)$, we can rewrite (7) as

$$H=-\lambda\sum_{\alpha,\beta=1}^{3}\sum_x(z^\beta C_\alpha^\dagger(x+1)C_\beta(x)+\mathrm{HC})-\sum_{\alpha=1}^{3}\sum_x(z^{\alpha-1}+z^{*\alpha-1})C_\alpha^\dagger(x)C_\alpha(x). \tag{11}$$

In order to take the naive continuum limit, we first rearrange terms as follows. Write

$$\sum_x\Gamma_1^\dagger(x+1)\Gamma_2(x)=\tfrac{1}{2}\sum_x\Gamma_1^\dagger(x+1)\Gamma_2(x)+\tfrac{1}{2}\sum\Gamma_1^\dagger(x)\Gamma_2(x-1) \tag{12}$$

where we have neglected a surface term. Next add and subtract the contribution

$$\lambda z\sum_x\Gamma_2^\dagger(x)\Gamma_1(x)+\lambda z^*\sum_x\Gamma_1^\dagger(x)\Gamma_2(x).$$

Remembering that $C_\alpha^\dagger(x)C_\beta(x)=0$ for $\alpha\neq\beta$, we can rewrite (11) in the form

$$\begin{aligned}H=&-\lambda\sum_x\sum_{\alpha,\beta=1}^{3}z^{*\alpha}C_\alpha^\dagger(x)\Delta_xC_\beta(x)+\lambda\sum_x\sum_{\alpha,\beta=1}^{3}z^\beta C_\alpha^\dagger(x)\Delta_x^-C_\beta(x)\\&-(1+\lambda z)\sum_x\sum_{\alpha,\beta}z^{\beta-1}C_\alpha^\dagger(x)C_\beta(x)-(1+\lambda z^*)\sum_x\sum_{\alpha,\beta}z^{*\alpha-1}C_\alpha^\dagger(x)C_\beta(x)\end{aligned} \tag{13}$$

where $\Delta_xf(x)\equiv f(x+1)-f(x)$ and $\Delta_x^-\equiv f(x)-f(x-1)$. We now use the property $C_1^\dagger(x)C_2^\dagger(x)=C_3(x)$ to simplify the last term a little. Also, we set $\lambda=1$ in the second term of (13) which amounts to neglecting an infinite constant. We now have

$$\begin{aligned}\frac{H}{\lambda}=&\sum_x\sum_{\alpha,\beta=1}^{3}(z^{*\alpha}C_\alpha^\dagger(x)\Delta_xC_\beta(x)-z^\beta C_\alpha^\dagger(x)\Delta_x^-C_\beta(x))\\&+\lambda^{-1}\sum_x\sum_{\alpha,\beta=1}^{3}(z^{\alpha+1}+z^{*\alpha+1})C_\alpha^\dagger(x)C_\alpha(x).\end{aligned} \tag{14}$$

We can now proceed to take the naive continuum limit by defining

$$C_\alpha(x)=\sqrt{a}\,\psi_\alpha(x) \qquad (\alpha=1,2). \tag{15}$$

In that case, we can drop all terms involving (say) C_3 and $C_3^\dagger$ since, being composed of the other two, they are of higher order in a and will scale out of the problem when we take $a\to0$. We also write the difference operator $\Delta_x=a\partial_x$ and take the limit $a\to0$ such that $\lambda a=\kappa^{-1}$ is held fixed. After these manipulations, we get

$$H=\int\mathrm{d}x\,H(x),$$

where

$$H(x)=-\sum_{\alpha,\beta=1}^{2}(z^{*\alpha}\psi_\alpha^\dagger(x)\partial_x\psi_\beta(x)-z^\beta\psi_\alpha^\dagger(x)\partial_x\psi_\beta(x))-\kappa\sum_{\alpha=1}^{2}\psi_\alpha^\dagger(x)\psi_\alpha(x). \tag{16}$$

The Hamiltonian (16) is by construction conformally invariant and we conjecture that $\kappa = 1$ corresponds to the true continuum limit of the Z_3 Potts model. The Hamiltonian (16) is completed by the commutation relations

$$\begin{aligned} &\psi^{\dagger}_{\alpha}(x)\psi_{\beta}(y) = (z\theta(x-y) + z^{*}\theta(y-x))\psi_{\beta}(y)\psi^{\dagger}_{\alpha}(x), \qquad x \neq y, \\ &\psi^{2}_{\alpha}(x) = 0 = \psi^{\dagger 2}_{\alpha}(x). \end{aligned} \tag{17}$$

Bosonisation of the model (16), (17) may lend further insight to its critical properties and complete solution.

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Note added in proof. A similar fermionisation scheme, in a different context, has also been obtained independently by Truang and de Vega (Paris Preprint).

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